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Coupling integrable field theories to mechanical systems at the boundary

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Abstract
We present an integrable Hamiltonian which describes the sinh-Gordon model on the half line coupled to a non-linear oscillator at the boundary. We explain how we apply Sklyanin's formalism to a dynamical reflection matrix to obtain this model. This method can be applied to couple other integrable field theories to dynamical systems at the boundary. We also show how to find the dynamical solution of the quantum reflection equation corresponding to our particular example.

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1. Introduction

Integrable field theories in two dimensions provide us with a theoretical laboratory to study non-perturbative phenomena in high-energy physics. They also possess numerous applications in fluid dynamics, statistical physics, condensed matter physics and quantum optics.

In recent years, it has become possible to extend many of the techniques and results of integrable models to field theories with boundaries, i.e. theories defined either on the half line or on an interval, see for instance [4, 6, 13, 23, 24].

We introduce a new class of integrable field theories with boundaries where, instead of imposing fixed boundary conditions, we couple the boundary field to a mechanical system.

We begin by giving a very concrete example in section 2 where we describe the sinh-Gordon or sine-Gordon model coupled to a non-linear oscillator at the boundary. Then in section 3 we explain more generally how one can couple an integrable field theory to a mechanical system in such a way that the integrability is not broken. We use a generalization of Sklyanin's technique for the construction of integrable boundary conditions. The new ingredient is that the solution of the reflection equation is chosen to depend on boundary degrees of freedom. In section 4 we specialize this technique to the case of the sine-Gordon theory providing some details of how we arrived at the model described in section 2. Finally, in section 5 we describe how to obtain a particular dynamical solution of the quantum reflection equation. This is the solution whose classical limit went into the construction in the previous section. We end with discussions in section 6.

Before we begin let us survey other works that deal with coupling to degrees of freedom at the boundaries. These fall into three categories:

- *Field theory.* In [3] the sine-Gordon model is coupled at the quantum level to a q-oscillator. In [21] it was observed that in order to derive the fixed boundary conditions for the supersymmetric sine-Gordon model from an action, it was convenient to introduce a fermionic variable at the boundary which could, however, be integrated out again immediately. In [12] a free fermion field theory is coupled to a dynamical boundary degree of freedom.
- *Integrable quantum spin chains.* Chains coupled to extra spins on the boundary are used in the study of Kondo impurities coupled to strongly correlated electron systems, see e.g. [28]. In [11, 30] new dynamical solutions of the quantum reflection equation that are not of the RKR type were used to describe the coupling of the boundary spins.
- *Systems with a finite number of degrees of freedom.* Kuznetsov [17] has used dynamical solutions of the classical reflection equation to couple mechanical tops to integrable non-linear lattices such as the Heisenberg chain or Toda lattices. Our work can be seen as the extension of these ideas to field theory while preserving integrability.

2. The sinh-Gordon model coupled to an oscillator at the boundary

In this section we present an integrable Hamiltonian describing the coupling of the sinh-Gordon field theory to a non-linear oscillator at the boundary. We leave the details of how we constructed this Hamiltonian for later sections.

The sinh-Gordon model describes a relativistic 1+1 dimensional self-interacting massive bosonic field $\phi(x, t)$. The Hamiltonian of the sinh-Gordon model restricted to the half-line is

$$H_{\text{shG}} = \int_{-\infty}^{0} \mathrm{d}x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 + \frac{m^2}{\hat{\beta}^2}(\cosh \hat{\beta}\phi - 1) \right). \tag{1}$$

Here π is the conjugate momentum to the field ϕ, i.e.

$$\{\pi(x), \phi(x)\} = \delta(x - y). \tag{2}$$

$\hat{\beta}$ is the real sinh-Gordon coupling constant and m sets the mass scale. We let space be the half line from $x = -\infty$ to $x = 0$. As usual the field ϕ is assumed to vanish at $x = -\infty$ but not at $x = 0$. As will become clear from the construction in section 3 we could also have taken space to be an interval $[x_-, x_+]$ and placed a mechanical system at both ends.

We describe the coupling of the sinh-Gordon field $\phi(0)$ at the boundary $x = 0$ to a non-linear oscillator through the boundary Hamiltonian

$$H_{\text{osc}} = \frac{2m}{\hat{\beta}^2} \left(\cosh\left(\frac{\hat{\beta}}{\sqrt{2Mm}}\, p \right) \mathrm{e}^{-\hat{\beta}\phi(0)/2} + \cosh\left(\frac{\hat{\beta}\sqrt{Mm}}{2\sqrt{2}}\, q \right) \mathrm{e}^{\hat{\beta}\phi(0)/2} \right). \tag{3}$$

Here q and p are the position and momentum variables of the oscillator and they obey the canonical Poisson bracket relation $\{p, q\} = 1$. The new free parameter M determines the mass of the oscillator.

It is instructive to expand the Hamiltonian for the oscillator for small p and q. One obtains

$$H_{\text{osc}} = \frac{1}{2M_\phi} p^2 + \frac{M_\phi}{2}\omega^2 q^2 + \frac{4m}{\hat{\beta}^2} \cosh\left(\frac{\hat{\beta}\phi(0)}{2} \right) + \mathcal{O}(p^4) + \mathcal{O}(q^4) \tag{4}$$

where

$$M_\phi = M \mathrm{e}^{\hat{\beta}\phi(0)/2} \qquad \text{and} \qquad \omega = m/2. \tag{5}$$

We see that the frequency ω of the oscillator is fixed by the requirement of integrability to be equal to half the mass of the sinh-Gordon field. The effective mass of the oscillator depends on the value of the sinh-Gordon field at the boundary. The exact form of the higher non-linear terms in the Hamiltonian of the oscillator is fixed by integrability.

To shorten the formulas we introduce the rescaled variables

$$\hat{p} = \frac{\hat{\beta}}{\sqrt{2Mm}} p \qquad \text{and} \qquad \hat{q} = \frac{\hat{\beta}\sqrt{Mm}}{2\sqrt{2}} q \tag{6}$$

which have Poisson bracket $\{\hat{p}, \hat{q}\} = \hat{\beta}^2/4$. In terms of these, the boundary equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p} = \{H_{\mathrm{osc}}, \hat{p}\} = -\frac{m}{2}\mathrm{e}^{\hat{\beta}\phi(0)/2}\sinh\hat{q} \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{q} = \{H_{\mathrm{osc}}, \hat{q}\} = \frac{m}{2}\mathrm{e}^{-\hat{\beta}\phi(0)/2}\sinh\hat{p}. \tag{8}$$

To determine the equations of motion of the sinh-Gordon field ϕ one needs to use the full Hamiltonian

$$H = H_{\mathrm{shG}} + H_{\mathrm{osc}}. \tag{9}$$

One finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(x) = \{H, \phi(x)\} = \pi(x) \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(x) = \{H, \pi(x)\} = \partial_x^2\phi(x) - \frac{m^2}{\hat{\beta}}\sinh\hat{\beta}\phi(x) - \delta(x)\left(\partial_x\phi(0) - \frac{m}{\hat{\beta}}\left(\mathrm{e}^{-\hat{\beta}\phi(0)/2}\cosh\hat{p} - \mathrm{e}^{\hat{\beta}\phi(0)/2}\cosh\hat{q}\right)\right). \tag{11}$$

Note the term proportional to $\delta(x)$ in the equation of motion for $\pi(x)$. It has two sources: the $\partial_x\phi$ arises when one performs a partial integration in the sinh-Gordon Hamiltonian which produces a boundary contribution at $x = 0$ and the other two terms arise because the Hamiltonian of the oscillator contains $\phi(0)$.

We are only going to allow solutions to these equations that are continuous on the left half-line $[-\infty, 0]$. So in particular we require continuity of $\pi(x)$ at $x = 0$. This implies that the $\delta(x)$ term in the equation of motion for $\pi(x)$ vanishes because this term would otherwise force $\pi(x)$ to develop a discontinuity at $x = 0$. We therefore know that the solutions will have to satisfy the boundary condition

$$\partial_x\phi(0) = \frac{m}{\hat{\beta}}\left(\cosh\hat{p}\,\mathrm{e}^{-\hat{\beta}\phi(0)/2} - \cosh\hat{q}\,\mathrm{e}^{\hat{\beta}\phi(0)/2}\right). \tag{12}$$

We can combine the equations of motion (10) and (11) to obtain the usual sinh-Gordon equation of motion

$$\partial_x^2\phi - \partial_t^2\phi = \frac{m^2}{\hat{\beta}}\sinh\hat{\beta}\phi. \tag{13}$$

This equation for ϕ together with the boundary condition (12) and the equations of motion (7) and (8) for the oscillator form one system of coupled equations involving both ODEs and a PDE that needs to be solved to obtain the time evolution of the system.

We call this system 'integrable' because, as we will show in section 4, we can construct an infinite number of conserved higher spin charges I_n that are in involution with each other. The first non-trivial charge beyond the Hamiltonian is the spin 3 charge

$$I_3 = \int_{-\infty}^{0} \left(\frac{\hat\beta^4}{16m^3} \left(\pi^4 + 6\pi^2(\partial_x\phi)^2 + (\partial_x\phi)^4\right) + \frac{\hat\beta^2}{m^3}\left((\partial_x\pi)^2 + \left(\partial_x^2\phi\right)^2\right) \right.$$
$$\left. + \frac{\hat\beta^2}{4m}\left(\pi^2 + 5(\partial_x\phi)^2\right)\cosh\hat\beta\phi + \frac{m}{8}(\cosh 2\hat\beta\phi - 1)\right) \mathrm{d}x$$
$$+ \mathrm{e}^{3\hat\beta\phi(0)/2}\left(\frac{1}{2}\cosh\hat q + \frac{1}{6}\cosh^3\hat q\right)$$
$$- \mathrm{e}^{\hat\beta\phi(0)/2}\left(\frac{3}{2}\cosh\hat p - \frac{1}{2}\cosh^2\hat q\cosh\hat p - \frac{\hat\beta^2}{2m^2}\pi^2\cosh\hat q\right)$$
$$+ \mathrm{e}^{-3\hat\beta\phi(0)/2}\left(\frac{1}{2}\cosh\hat p + \frac{1}{6}\cosh^3\hat p\right)$$
$$- \mathrm{e}^{-\hat\beta\phi(0)/2}\left(\frac{3}{2}\cosh\hat q - \frac{1}{2}\cosh^2\hat p\cosh\hat q - \frac{\hat\beta^2}{2m^2}\pi^2\cosh\hat p\right)$$
$$+ \frac{2\hat\beta}{m}\sinh\hat q\sinh\hat p. \tag{14}$$

The boundary condition (12) is similar in form to the previously known integrable boundary conditions [13, 19]

$$\partial_x\phi(0) = \frac{m}{\hat\beta}\left(\epsilon_0\,\mathrm{e}^{-\hat\beta\phi(0)/2} - \epsilon_1\,\mathrm{e}^{\hat\beta\phi(0)/2}\right) \tag{15}$$

with the crucial difference of course that the parameters ϵ_0 and ϵ_1 were fixed numbers rather than dynamical variables as in our case. Only in one case do our boundary conditions reduce to the fixed boundary conditions (15), namely when the boundary oscillator is at rest, i.e. at $q = p = 0$. This corresponds to the boundary conditions with $\epsilon_0 = \epsilon_1 = 1$. The quantum fluctuations of the boundary oscillator will probably imply that our model never reduces to the usual boundary conditions in the quantum case.

After quantization the sinh-Gordon model describes scalar massive particles. The direct solution of the quantum theory is rather difficult [25]. However, because of the existence of higher spin conserved charges one knows that there is no particle production and that the particle scattering factorizes into a product of two-particle scattering processes. The corresponding scattering amplitude has been obtained by analytical continuation in the coupling constant from the breather scattering amplitude in the sine-Gordon model [29].

To describe the sinh-Gordon particles on the half-line one also needs to give the reflection amplitudes. In the case of the fixed boundary conditions (15) this amplitude can again be obtained from the corresponding breather reflection amplitude in the sine-Gordon model [14]. Because we expect that the results for our model can be obtained similarly, we now turn our attention to the sine-Gordon model.

When we let the sinh-Gordon coupling constant $\hat\beta$ become purely imaginary, i.e. if we set $\hat\beta = \mathrm{i}\beta$ with β purely real, then the sinh-Gordon Hamiltonian (1) turns into the sine-Gordon Hamiltonian

$$H_{\mathrm{sG}} = \int_{-\infty}^{0} \mathrm{d}x\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2 - \frac{m^2}{\beta^2}(\cos\beta\phi - 1)\right). \tag{16}$$

Under the same replacement the Hamiltonian for the oscillator becomes

$$H_{\mathrm{osc}} = -\frac{2m}{\beta^2}\left(\cos\left(\frac{\beta}{\sqrt{2Mm}}p\right)\mathrm{e}^{-\mathrm{i}\beta\phi(0)/2} + \cos\left(\frac{\beta\sqrt{Mm}}{2\sqrt{2}}q\right)\mathrm{e}^{\mathrm{i}\beta\phi(0)/2}\right). \tag{17}$$

Note that in general this Hamiltonian is not real. We are familiar with this situation from imaginary coupled affine Toda theories. There the non-real Hamiltonian implies that the classical soliton solutions are complex. The energy of these configurations is nevertheless real [15]. Similarly, here the classical boundary solutions will be complex, but we expect that their energies will be real.

3. Sklyanin's formalism

In a seminal two-page paper [23] Sklyanin described how one can impose boundary conditions on an integrable field theory without breaking integrability. Below we will describe how Sklyanin's formalism allows us to couple an integrable field theory to a mechanical system at the boundary rather than imposing a fixed boundary condition. This section will be easier to understand for readers who are familiar with the approach to integrable models described in [10].

In the following we assume that there exists a pair of matrix valued functions $a_x(\lambda, x)$ and $a_t(\lambda, x)$ which depend on the fields of the theory, their conjugate momenta, and on a spectral parameter $\lambda = \mathrm{e}^{\theta} \in \mathbb{C}$, so that the classical equations of motion of the field theory are equivalent to the Lax pair equation

$$[\partial_x - a_x(\lambda, x), \partial_t - a_t(\lambda, x)] = 0 \qquad \text{for all } \lambda. \tag{18}$$

Here $a_x(\lambda, x)$ and $a_t(\lambda, x)$ depend on x and t only implicitly through their dependence on the fields. ∂_x and ∂_t denote total differentiations with respect to the space or time variable. If $a_x(\lambda, x)$ and $a_t(\lambda, x)$ are thought of as the components of a connection then equation (18) is the zero curvature condition for this connection.

Equation (18) is the compatibility condition for the overdetermined system of equations

$$\begin{aligned}\frac{\partial T}{\partial x_+} &= a_x(\lambda, x_+)T \\ \frac{\partial T}{\partial t} &= a_t(\lambda, x_+)T - T a_t(\lambda, x_-)\end{aligned} \tag{19}$$

where the transition matrix $\mathcal{T} \equiv T(x_+, x_-, \lambda)$ is defined to be a solution of the differential equations (19) with the initial conditions $T(x_-, x_-, \lambda) = I$. It can be expressed as the path-ordered exponential of $a_x(\lambda, x)$ from x_- to x_+

$$T(x_+, x_-, \lambda) = \mathcal{P}\exp\left(\int_{x_-}^{x_+} a_x(\lambda, x)\,\mathrm{d}x\right) \tag{20}$$

so that the operators at points nearer to x_+ are further to the left. We assume that the Poisson brackets for the functions $a_x(\lambda, x)$ can be written in the form

$$\left\{\overset{1}{a}_x(\lambda_1, x), \overset{2}{a}_x(\lambda_2, y)\right\} = \delta(x-y)\left[r(\ln(\lambda_1/\lambda_2)), \overset{1}{a}_x(\lambda_1, x) + \overset{2}{a}_x(\lambda_2, y)\right]. \tag{21}$$

Here we used the short-hand notations $\overset{1}{a}(\lambda, x) = a(\lambda, x) \otimes I$, $\overset{2}{a}(\lambda, x) = I \otimes a(\lambda, x)$ and the r-matrix is independent of the field or its conjugate momentum. It follows that

$$\left\{\overset{1}{T}(x_+, x_-, \lambda_1), \overset{2}{T}(x_+, x_-, \lambda_2)\right\} = \left[r(\ln(\lambda_1/\lambda_2)), \overset{1}{T}(x_+, x_-, \lambda_1)\overset{2}{T}(x_+, x_-, \lambda_2)\right]. \tag{22}$$

For simplicity we assume below that the r-matrix has the property[1]

$$r(\theta) = -r(-\theta). \tag{23}$$

Let us now introduce two matrix valued functions $K_\pm(\theta)$ of the spectral parameter $\lambda = e^\theta$, not depending on the fields. However, in departure from the situation described by Sklyanin in [23], we let these boundary K matrices be dynamical. This means that we enlarge the phase space of the theory by introducing extra degrees of freedom which we think of as describing mechanical systems placed at the boundaries. We then let the K matrices depend on these new dynamical variables in such a way that their Poisson brackets are given by the following classical reflection equation:

$$\left\{ \overset{1}{K}_\pm(\theta), \overset{2}{K}_\pm(\theta') \right\} = \left[r(\theta-\theta'), \overset{1}{K}_\pm(\theta)\overset{2}{K}_\pm(\theta') \right] + \overset{1}{K}_\pm(\theta) r(\theta+\theta') \overset{2}{K}_\pm(\theta') - \overset{2}{K}_\pm(\theta') r(\theta+\theta') \overset{1}{K}_\pm(\theta). \tag{24}$$

Furthermore, we assume that the dynamical systems on the left and right boundary are independent so that $\left\{ \overset{1}{K}_\pm(\theta), \overset{2}{K}_\mp(\theta') \right\} = 0$.

Following Sklyanin [23] we define the functional

$$\mathcal{T}(x_+, x_-, \lambda) = T(x_+, x_-, \lambda) K_-(\ln\lambda) T^{-1}(x_+, x_-, 1/\lambda) \tag{25}$$

which generalizes the transition matrix to the boundary case. Using equation (23) and that the Poisson bracket is antisymmetric, satisfies the Jacobi identity and has the property $\{A, BC\} = \{A, B\}C + B\{A, C\}$ it is straightforward to show that $\mathcal{T}(x_+, x_-, \lambda)$ obeys a Poisson bracket relation similar to those for the K's given in equation (24). A family of transfer matrices is defined by

$$\tau(\lambda) = \mathrm{tr}\big(K_+(\ln(\lambda))\mathcal{T}(x_+, x_-, \lambda)\big). \tag{26}$$

These are in involution for any values of the spectral parameters

$$\{\tau(\lambda_1), \tau(\lambda_2)\} = 0 \qquad \text{for all } (\lambda_1, \lambda_2) \in \mathbb{C} \tag{27}$$

provided appropriate boundary conditions are imposed at $x_\pm$. As the generating function $\tau(\lambda)$ can be expanded about the singularities in the transition matrix $T(x_+, x_-, \lambda)$ it gives an infinite number of quantities I_n in involution with each other. Among these we will identify the Hamiltonian of the model. Then it follows that the $\{I_n\}$ are time conserved. The model is thus integrable on the interval $[x_-, x_+]$. Different dynamical systems can be coupled at the two boundaries by choosing different K_- and K_+.

4. Derivation of the sine-Gordon example

In this section we will provide some details needed to apply the general method described in the previous section to the sine-Gordon model in order to derive the coupling to a boundary oscillator described in section 2.

The equation of motion (13) for the sine-Gordon field is representable as the zero-curvature condition for the Lax connection $a_\mu(\lambda, x)$ written in terms of the standard Pauli matrices σ_k, $k = 1, 2, 3$ as

$$a_x(\lambda, x) = \frac{\beta}{4i}\frac{\partial\phi}{\partial t}\sigma_3 + \frac{m}{4i}\left(\lambda + \frac{1}{\lambda}\right)\sin\left(\frac{\beta\phi}{2}\right)\sigma_1 + \frac{m}{4i}\left(\lambda - \frac{1}{\lambda}\right)\cos\left(\frac{\beta\phi}{2}\right)\sigma_2 \tag{28}$$

$$a_t(\lambda, x) = \frac{\beta}{4i}\frac{\partial\phi}{\partial x}\sigma_3 + \frac{m}{4i}\left(\lambda - \frac{1}{\lambda}\right)\sin\left(\frac{\beta\phi}{2}\right)\sigma_1 + \frac{m}{4i}\left(\lambda + \frac{1}{\lambda}\right)\cos\left(\frac{\beta\phi}{2}\right)\sigma_2. \tag{29}$$

[1] This could however be relaxed, see for example [9] for an application of Sklyanin's formalism to affine Toda theory where the r-matrix does not possess this property.

We define $\pi(x,t) = \frac{\partial\phi(x,t)}{\partial t}$. Note that the matrix $a_x(\lambda, x)$ possesses two singularities located at $|\lambda| = 0$ and $|\lambda| = \infty$.

To extend the standard definition of the Poisson bracket, we change the range of the integrals from $[-\infty, +\infty]$ to $[x_-, x_+]$. Then, using the notation $\pi(x) \equiv \pi(x, t)$ and $\phi(x) \equiv \phi(x, t)$ at fixed time t we define the Poisson bracket as

$$\{\mathcal{O}_1, \mathcal{O}_2\} = \int_{x_-}^{x_+} \mathrm{d}x \left[\frac{\delta\mathcal{O}_1}{\delta\pi(x)}\frac{\delta\mathcal{O}_2}{\delta\phi(x)} - \frac{\delta\mathcal{O}_1}{\delta\phi(x)}\frac{\delta\mathcal{O}_2}{\delta\pi(x)}\right] + \frac{\partial\mathcal{O}_1}{\partial p}\frac{\partial\mathcal{O}_2}{\partial q} - \frac{\partial\mathcal{O}_1}{\partial q}\frac{\partial\mathcal{O}_2}{\partial p} \tag{30}$$

for any observable $\mathcal{O}_j$. Obviously, this bracket possesses the basic properties of a Poisson bracket. It is skew symmetric and satisfies the Jacobi identity. Also, we have $\{\mathcal{O}_1, \mathcal{O}_2\mathcal{O}_3\} = \{\mathcal{O}_1, \mathcal{O}_2\}\mathcal{O}_3 + \mathcal{O}_2\{\mathcal{O}_1, \mathcal{O}_3\}$. Using the definition (30), the non-vanishing Poisson brackets in the sine-Gordon field theory at constant time slices are $\{\pi(x), \phi(y)\} = \delta(x - y)$ and for the boundary variables $p(t)$, $q(t)$ we have $\{p, q\} = 1$. Now, by calculating the Poisson brackets of the entries of the matrix $a_x(\lambda, x)$ it gives a unique solution to equation (22):

$$r_{SG}(\theta) = \frac{\beta^2\cosh(\theta)}{16\sinh(\theta)}(I\otimes I - \sigma_3\otimes\sigma_3) - \frac{\beta^2}{16\sinh(\theta)}(\sigma_1\otimes\sigma_1 + \sigma_2\otimes\sigma_2). \tag{31}$$

To construct the generating function (26) of the integrals of motion for the sine-Gordon model with dynamical boundaries, we need (p, q)-dependent solutions $K_\pm(\theta)$ to the classical reflection equation (24). Let us introduce the rescaled variables $\tilde{p} = p\beta/2\sqrt{2}$ and $\tilde{q} = q\beta/2\sqrt{2}$. Then the matrix [26]

$$K_+(\theta) = 2\begin{pmatrix}\cosh(\tilde{p}+\tilde{q})\mathrm{e}^{\theta} - \cosh(\tilde{p}-\tilde{q})\mathrm{e}^{-\theta} & 2\sinh^2(\theta) - 2\sinh^2(\tilde{p}) \\ 2\sinh^2(\tilde{q}) - 2\sinh^2(\theta) & \cosh(\tilde{p}-\tilde{q})\mathrm{e}^{\theta} - \cosh(\tilde{p}+\tilde{p})\mathrm{e}^{-\theta}\end{pmatrix} \tag{32}$$

satisfies the classical reflection equation (24) with the classical r-matrix $r(\theta) = r_{SG}(\theta)$ defined above. We could place additional degrees of freedom at the left boundary as well and use a K_- of a similar form. However we would not learn anything new and for simplicity we choose $K_-(\theta) = I$, which is a trivial solution of the classical reflection equation (24) and then move the left boundary off to $-\infty$, i.e. we restrict the sine-Gordon field theory to the half-line. We then assume that the field and its conjugate momentum satisfy the Schwartz boundary condition $\phi(x_-, t) = 0$ and $\pi(x_-, t) = 0$ at $x_- = -\infty$. Using equation (25) the generating function (26) becomes

$$\tau(\lambda) = \mathrm{tr}\big(K_+(\ln(\lambda))T(0, -\infty, \lambda)T(-\infty, 0, 1/\lambda)\big). \tag{33}$$

Fixing $\mathrm{Im}(\theta) = \frac{\pi}{2}$, the transition matrix $T(-\infty, 0, \lambda)$ has singularities located at $\lambda = \pm\mathrm{i}\infty$ and $\lambda = \pm\mathrm{i}0$. There will exist two infinities of involutive integrals coming from the coefficients of the Laurent expansions about these two values. First, let us consider the asymptotic expansion of the transfer matrix $\tau(\lambda)$ as $|\lambda| \to \infty$. Substituting the solution $K_+(\theta)$ of (32) into (33) and expanding about $\lambda = \mathrm{i}\infty$, we obtain a Laurent series in λ which provides an infinite number of quantities I_n:

$$\ln\left(\tau(\lambda)/\lambda^2\right) = \sum_{n=-1}^{\infty}\frac{I_n}{\lambda^n}. \tag{34}$$

In performing this expansion it has helped us to look at how the corresponding calculation was performed in [19] for the case of non-dynamical K. We find $I_{-1} = -\frac{\mathrm{i}mL}{2}|_{L\to\infty}$ and $I_0 = 0$. The next quantity $I_1 = -\frac{\mathrm{i}\beta^2}{2m}H + \mathrm{const}$ gives the Hamiltonian H for the system, where

$$H = \int_{-\infty}^{0}\mathrm{d}x\left[\Theta(-x)\left(\frac{1}{2}\left[\pi^2 + (\partial_x\phi)^2\right] - \frac{m^2}{\beta^2}(\cos(\beta\phi) - 1)\right) + \delta(x)\frac{2m}{\beta^2}\left(\cosh(\tilde{p}+\tilde{q})\mathrm{e}^{\mathrm{i}\beta\phi/2} + \cosh(\tilde{p}-\tilde{q})\mathrm{e}^{-\mathrm{i}\beta\phi/2}\right)\right]. \tag{35}$$

This can be seen to agree with the Hamiltonian given in section 2 after we perform the canonical transformation

$$\tilde{p} + \tilde{q} \to \hat{q} \quad \tilde{p} - \tilde{q} \to -\hat{p} \quad \phi(x) \to \phi(x) + 2\pi/\beta. \tag{36}$$

The next term in the expansion (34) of order $O(1/\lambda^2)$ reduces to $I_2 = 0$ after using the boundary condition (12). Similarly the $O(1/\lambda^3)$ term in (34) reproduces the spin 3 charge I_3 given in (14), again after the canonical transformation (36). It is believed that the existence of such higher local integrals of motion is a sufficient condition for the classical integrability of the system.

Finally we observe that because the transfer matrix $\tau(\lambda)$ in (33) is invariant under the simultaneous replacements $\lambda \to -1/\lambda$, $\phi(x,t) \to -\phi(x,t)$ and $q \to -q$, the expansion of $\tau(\lambda)$ around the singularity at 0 gives analogous quantities I_n. Note that H and I_3 are invariant under this transformation.

It is rather interesting to note the following: if p and q are fixed c-numbers then, the (p,q)-dependent part of the Poisson bracket (30) disappears and the left-hand side of equation (24) vanishes. Considering the two leading terms in the expansion in λ of the classical reflection matrix $K_+(\theta)$ given in equation (32) we obtain

$$K_+(\ln(\lambda)) \underset{|\lambda|\to\infty}{=} \lambda^2 \begin{pmatrix} \frac{2}{\lambda}\cosh(\tilde{p}+\tilde{q}) & 1 \\ -1 & \frac{2}{\lambda}\cosh(\tilde{p}-\tilde{q}) \end{pmatrix} + O(\lambda^0) \tag{37}$$

which can be shown to satisfy equation (24). Let us now introduce the notation

$$\begin{aligned} \cosh(\tilde{p}+\tilde{q}) &= P + \mathrm{i}Q \\ \cosh(\tilde{p}-\tilde{q}) &= P - \mathrm{i}Q. \end{aligned} \tag{38}$$

Multiplying $K_+(\ln(\lambda))$ by $1/\lambda^2$ the reflection matrix now writes

$$K_+(\ln(\lambda)) \sim \frac{2P}{\lambda} I + \mathrm{i}\sigma_2 + \frac{2\mathrm{i}Q}{\lambda}\sigma_3 + O(1/\lambda^2)\,. \tag{39}$$

Up to order $1/\lambda^2$ this is the same K-matrix used in [19] to construct the integrable boundary conditions used by Ghoshal and Zamolodchikov in [13]. Of course, our next quantities I_2 and I_3 differ from those in [19] due to the contributions of order $O(1/\lambda^2)$ that appear in our case.

5. Dynamical solutions of the quantum reflection equation

In this section we look for operator-valued solutions $\mathcal{K}_\pm(\theta;\alpha)$ of the quantum reflection equations [24][2]

$$\begin{aligned} R(\theta-\theta')\overset{1}{\mathcal{K}}_-(\theta+\alpha/2;\alpha)R(\theta+\theta')\overset{2}{\mathcal{K}}_-(\theta'+\alpha/2;\alpha) \\ = \overset{2}{\mathcal{K}}_-(\theta'+\alpha/2;\alpha)R(\theta+\theta')\overset{1}{\mathcal{K}}_-(\theta+\alpha/2;\alpha)R(\theta-\theta') \end{aligned} \tag{40}$$

$$\begin{aligned} R(-\theta+\theta')\overset{1}{\mathcal{K}}{}^t_+(\theta-\alpha/2;\alpha)R(-\theta-\theta')\overset{2}{\mathcal{K}}{}^t_+(\theta'-\alpha/2;\alpha) \\ = \overset{2}{\mathcal{K}}{}^t_+(\theta'-\alpha/2;\alpha)R(-\theta-\theta')\overset{1}{\mathcal{K}}{}^t_+(\theta-\alpha/2;\alpha)R(-\theta+\theta'). \end{aligned} \tag{41}$$

Here $R(\theta)$ is a given quantum R-matrix. We see that the quantum reflection matrix $\mathcal{K}_-(\theta;\alpha)$ for the left boundary has to satisfy a different equation to the quantum reflection matrix $\mathcal{K}_+(\theta;\alpha)$ for the right boundary. The classical reflection equation (24) can be obtained as a limiting case of either of the quantum reflection equations (40) and (41)[3]

[2] Our $\mathcal{K}_\pm$ are related to the $K_\pm$ of [24] by $\mathcal{K}_\pm(\theta+\alpha/2) = K_\pm(\theta)$ and $\alpha = \eta$.

[3] In the limit $\alpha \to 0$ one recovers the classical R-matrix (up to an overall θ-dependent coefficient) $R(\theta) \sim I + \alpha r(\theta) + O(\alpha^2)$ where I is the identity matrix. The classical reflection matrices $K_\pm(\theta)$ are defined to be the first term in the expansion of the quantum reflection matrices: $\mathcal{K}_\pm(\theta;\alpha) \underset{\alpha\to 0}{=} K_\pm(\theta) + \alpha\delta K_\pm(\theta) + O(\alpha^2)$.

One useful way to think of the reflection equation (40) is to view the entries of the matrix $\mathcal{K}_-(\theta;\alpha)$ as the generators of an associative algebra with quadratic algebra relations given by (40). Such an algebra is often called a reflection equation algebra. Finding a solution of a reflection equation is therefore the same as finding a representation of the corresponding reflection equation algebra. In equation (46) we will give an infinite dimensional representation in terms of position and momentum operators q and p.

Operator-valued reflection matrices have several applications: (1) they can be used in integrable lattice models or quantum spin chains to couple additional boundary spins to the model [11, 30], (2) they can describe the reflection of particle excitations from a boundary if there are degeneracies in the boundary spectrum, and (3) their classical limit can be used to construct integrable models [17, 26].

As an example, let us consider the following trigonometric and hyperbolic minimal solutions of the quantum Yang–Baxter equation:

$$R(\theta,\gamma) = \frac{a(\theta)}{2}\,(I\otimes I + \sigma_3\otimes\sigma_3) + \frac{b(\theta)}{2}\,(I\otimes I - \sigma_3\otimes\sigma_3) + \frac{c(\theta)}{2}\,(\sigma_1\otimes\sigma_1 + \sigma_2\otimes\sigma_2)\,. \tag{42}$$

This R-matrix gives the Boltzmann weights of the six-vertex model which is known to be related to the XXZ and the XXX (in its rational limit) spin chains. It possesses three different regimes called antiferroelectric (I), trigonometric (II) and ferroelectric (III) with

(I)	$a(\theta)=\sinh(\gamma-\theta)$	$b(\theta)=\sinh(\theta)$	$c(\theta)=\sinh(\gamma)$	$\gamma>\theta>0$
(II)	$a(\theta)=\sin(\gamma-\theta)$	$b(\theta)=\sin(\theta)$	$c(\theta)=\sin(\gamma)$	$\pi>\gamma>\theta>0$
(III)	$a(\theta)=\sinh(\theta+\gamma)$	$b(\theta)=\sinh(\theta)$	$c(\theta)=\sinh(\gamma)$	$\theta>0,\gamma>0$

where the parameter γ characterizes the anisotropy of the model. In particular, the trigonometric regime of the six-vertex model (II) describes the critical (zero gap) limit of the eight-vertex model [2].

A dynamic solution of the quantum reflection equation corresponding to this R-matrix was already given in [16, 18] without derivation. We will derive a similar solution below but with some minor quantum adjustments.

For further convenience, defining the K-matrix at the quantum level by

$$\mathcal{K}_-(\theta;\alpha) = \begin{pmatrix} A(\theta) & B(\theta) \\ D(\theta) & E(\theta) \end{pmatrix} \tag{43}$$

and using the expression for the R-matrix written above, the reflection equation (40) now reduces to eight functional equations:

$$\begin{aligned}
&\text{(i)} && a_-c_+(BD'-B'D) + a_-a_+[A,A'] = 0\\
&\text{(ii)} && b_-b_+(AE'-E'A) + c_-c_+[E,E'] + c_-a_+(DB'-D'B) = 0\\
&\text{(iii)} && a_-c_+(DB'-D'B) + a_-a_+[E,E'] = 0\\
&\text{(iv)} && c_-b_+(EA'-E'A) + b_-c_+(AA'-E'E) + b_-a_+(BD'-D'B) = 0\\
&\text{(v)} && b_-b_+AD' + c_-c_+ED' + c_-a_+DA' - a_-a_+D'A - a_-c_+E'D = 0\\
&\text{(vi)} && b_-a_+BE' + c_-b_+EB' + b_-c_+AB' - a_-b_+E'B = 0
\end{aligned}$$

where we use the shorthand notations $a_- = a(\theta-\theta')$, $a_+ = a(\theta+\theta'-\alpha)$ and similarly for b and c as well as $A = A(\theta)$ and $A' = A(\theta')$ and similarly for B, D and E. The remaining two equations are obtained from (v) and (vi) through the substitutions $A \leftrightarrow E$ and $B \leftrightarrow D$. Let us now focus on solutions of kind (III). To find the solution of the equations (i)–(vi), we assume the following form for the K-matrix:

$$A(\theta) = F\mathrm{e}^{\theta+\alpha/2} - G\mathrm{e}^{-\theta-\alpha/2} \qquad E(\theta) = G\mathrm{e}^{\theta+\alpha/2} - F\mathrm{e}^{-\theta-\alpha/2}$$
$$B(\theta) = -2\sinh^2\theta + 2U \qquad D(\theta) = 2\sinh^2\theta - 2V.$$

Setting $\gamma = \alpha$ in case (III), the generators $\{F, G, U, V\}$ have to satisfy the following relations:

$$\begin{aligned}
&[F, G] = -2\sinh\alpha(U - V) \qquad [U, V] = \frac{\sinh 2\alpha}{2}(F^2 - G^2)\\
&FV\mathrm{e}^{-\alpha} - VF\mathrm{e}^{\alpha} - F\sinh\alpha + G\sinh 2\alpha/2 = 0\\
&GV\mathrm{e}^{\alpha} - VG\mathrm{e}^{-\alpha} + G\sinh\alpha - F\sinh 2\alpha/2 = 0\\
&FU\mathrm{e}^{-\alpha} - VF\mathrm{e}^{\alpha} - F\sinh\alpha + G\sinh 2\alpha/2 = 0\\
&GU\mathrm{e}^{\alpha} - UG\mathrm{e}^{-\alpha} + G\sinh\alpha - F\sinh 2\alpha/2 = 0\,.
\end{aligned} \tag{44}$$

One can find a representation for the generators $\{F, G, U, V\}$ satisfying these quadratic-linear relations in terms of the position and momentum operators q and p with commutation relation

$$[p(t), q(t)] = \alpha. \tag{45}$$

This leads to the following solution of the quantum reflection equation (40):

$$\mathcal{K}_-(\theta;\alpha) = \begin{pmatrix} \cosh(p-q)\mathrm{e}^{\theta+\alpha/2} - \cosh(p+q)\mathrm{e}^{-\theta-\alpha/2} & 2\sinh^2(q) - 2\sinh^2(\theta) \\ 2\sinh^2(\theta) - 2\sinh^2(p) & \cosh(p+q)\mathrm{e}^{\theta+\alpha/2} - \cosh(p-q)\mathrm{e}^{-\theta-\alpha/2} \end{pmatrix}. \tag{46}$$

This $\mathcal{K}_-(\theta;\alpha)$ satisfies (40) also in regime (I) with $\gamma = -\alpha$. With the substitution $\theta \to \mathrm{i}\theta$ and $\gamma \to \mathrm{i}\gamma$ the solution corresponding to the regime (II) follows immediately. The matrix $\mathcal{K}_+(\theta;\alpha) = \mathcal{K}_-^t(-\theta;\alpha)$ provides a solution of (41). The classical reflection matrix (32) used in the previous section is obtained as the classical limit of $\mathcal{K}_-(\theta;\alpha)$.

As was observed by Sklyanin, given a c-number-valued solution K of the reflection equation one can always construct an infinite number of additional operator-valued solutions by dressing K with R-matrices to RKR, see [25] for details. The solution that we derived above is probably not of this factorizable form.

6. Discussion

We have explicitly constructed an integrable classical field theory with extra degrees of freedom living at the boundary. We have shown how to obtain concrete expressions for the Hamiltonian and higher spin-conserved charges.

While in this paper the sinh-Gordon model coupled to an oscillator at the boundary was given only as an example of the kind of model one can obtain, we believe that it is of great interest in its own right and we intend to study both its classical solutions and its quantization. In both endeavours we will be helped by the large amount of work that has been done already on the sine-Gordon model with fixed boundary conditions, see for example [1, 13, 20, 22, 27].

Of particular interest are the classical solutions describing oscillating boundary states. In the case of the fixed boundary conditions these so-called boundary breather solutions were found and quantized semiclassically to obtain the spectrum of boundary states [7, 8]. In our model we will look for classical solutions in which not only the field $\phi(x)$ near the boundary, but also the boundary variables p and q oscillate. It is possible that there will be two or more solutions with the same energy, they might for example be obtained from one another through the transformation $\phi \to -\phi$, $p \to -q$, $q \to p$, which is a symmetry of the theory. This question of how many degenerate states there are, is very relevant to the determination of the reflection matrices. As in the case of fixed boundary conditions described in [13], the soliton reflection matrices will have to satisfy the quantum reflection equation. However, if there is a degeneracy of boundary states then one will look for operator-valued solutions of the reflection equation so as to be able to describe processes where the reflection of a soliton rotates the degenerate boundary states among themselves.

The sinh-Gordon model is just the simplest member of the family of affine Toda field theories. For all Toda theories there exist integrable boundary conditions of the form [5]

$$\partial_x \vec{\phi}(0) = \sum_{i=0}^{r} A_i \vec{\alpha}_i \mathrm{e}^{\vec{\alpha}_i \cdot \vec{\phi}(0)/2} . \tag{47}$$

However, the boundary parameters A_i were found to be restricted by the requirement of integrability to only a small discrete set of values. It would be nice to see if one could replace the fixed parameters A_i by the dynamical variables of a mechanical system in the way in which we have done for the sinh-Gordon model in this paper. The fixed boundary conditions might then arise as the possible stationary points of the boundary system.

Besides the affine Toda field theories, whose integrability is based on the trigonometric R-matrices, there are many integrable field theories related to rational R-matrices, for example, the non-linear Schrödinger model and the principal chiral models. Finding dynamical solutions of the reflection equation for these rational models is simpler than in the trigonometric case and many are already known [17]. These could be used to construct integrable couplings to boundary mechanical systems for these field theories.

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References

[1] Bajnok Z, Palla L, Takacs G and Toth G Zs 2001 The spectrum of boundary states is sine-Gordon model with integrable boundary conditions *Preprint* hep-th/0106070

[2] Baxter R J 1982 *Exactly Solvable Models in Statistical Mechanics* (New York: Academic)

[3] Bazhanov V, Lukyanov S and Zamolodchikov A 1999 On nonequilibrium states in QFT model with boundary interaction *Nucl. Phys.* B **549** 529
(Bazhanov V, Lukyanov S and Zamolodchikov A 1998 *Preprint* hep-th/9812091)

[4] Cherednik I V 1984 Factorizing particles on a half line and root systems *Theor. Math. Phys.* **61** 997

[5] Corrigan E, Dorey P E, Rietdijk R H and Sasaki R 1994 Affine Toda field theory on a half line *Phys. Lett.* B **333** 83
(Corrigan E, Dorey P E, Retdiik R H and Sasaki R 1994 *Preprint* hep-th/9404108)

[6] Corrigan E 1998 *Frontiers in Quantum Field Theory* ed C-Z Zha and K Wu (Singapore: World Scientific) pp 9–32
(Corrigan E 1996 *Preprint* hep-th/9612138)

[7] Corrigan E and Delius G W 1999 Boundary breathers in the sinh-Gordon model *J. Phys. A: Math. Gen.* **32** 8601
(Corrigan E and Delius G W 1999 *Preprint* hep-th/9909145)

[8] Corrigan E and Taormina A 2000 Reflection factors and a two-parameter family of boundary bound states in the sinh-Gordon model *J. Phys. A: Math. Gen.* **33** 8739
(Corrigan E and Taormina A 2000 *Preprint* hep-th/0008237)

[9] Delius G W 1998 Soliton-preserving boundary conditions in affine Toda field theories *Phys. Lett.* B **444** 217
(Delius G W 1998 *Perprint* hep-th/9809140)

[10] Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)

[11] Frahm H and Slavnov N A 1999 New solutions to the reflection equation and the projecting method *J. Phys. A: Math. Gen.* **32** 1547
(Frahm H and Salvnov N A 1998 *Preprint* cond-mat/9810312)

[12] Fuentes M, Lopez A, Fradkin E and Moreno E 1995 Bosonization rules in $1/2 + 1$ dimensions *Nucl. Phys.* B **450** 603
(Fuentes M, Lopez A, Fradkin E and Moreno E 1995 *Preprint* cond-mat/9502076)

[13] Ghoshal S and Zamolodchikov A B 1994 Boundary *S* matrix and boundary state in two-dimensional integrable field theory *Int. J. Mod. Phys.* **9** 3841
(Ghoshal S and Zamolodchikov A B 1993 *Preprint* hep-th/9306002)

[14] Ghoshal S 1994 Bound state boundary *S*-matrix of the sine-Gordon model *Int. J. Mod. Phys.* A **9** 4801
(Ghoshal S 1993 *Preprint* hep-th/9310188)

[15] Hollowood T 1992 Solitons in affine Toda field theories *Nucl. Phys.* B **384** 523

[16] Kuznetsov V B and Tsiganov A V 1994 Separation of variables for the quantum relativistic Toda lattices *Preprint* hep-th/9402111

[17] Kuznetsov V B 1995 New Boundary Conditions for Integrable Lattices *J. Phys. A: Math. Gen.* **28** 4639
(Kuznetsov V B 1995 *Preprint* hep-th/9503168)

[18] Kuznetsov V B and Tsyganov A V 1996 Quantum relativistic Toda lattices *J. Math. Sci.* **80** 1802

[19] MacIntyre A 1995 Integrable boundary conditions for classical sine-Gordon theory *J. Phys. A: Math. Gen.* **28** 1089
(MacIntyre A 1994 *Preprint* hep-th/9410026)

[20] Mattsson P and Dorey P 2000 *Preprint* hep-th/0008071

[21] Nepomechie R I 2001 The boundary supersymmetric sine-Gordon model revisited *Phys. Lett.* B **509** 183
(Nepomechie R I 2001 *Preprint* hep-th/0103029)

[22] Saleur H, Skorik S and Warner N P 1995 The boundary sine-Gordon theory classical and semi-classical analysis *Nucl. Phys.* B **441** 421
(Saleur H, Skorik S and Warner N P 1994 *Preprint* hep-th/9408004)

[23] Sklyanin E K 1987 Boundary conditions for integrable equations *Funct. Anal. Appl.* **21** 164

[24] Sklyanin E K 1988 Boundary conditions for integrable quantum systems *J. Phys. A: Math. Gen.* **21** 2375

[25] Sklyanin E K 1989 Exact quantization of the sinh-Gordon model *Nucl. Phys.* B **326** 719

[26] Suris Yu B 1990 Discrete time generalized Toda lattices: complete integrability and relation with relativistic Toda lattices *Phys. Lett.* A **145** 113

[27] Tarasov V O 1991 The integrable initial-boundary value problem on a semiline: nonlinear Schrödinger and sine-Gordon equations *Inverse Problems* **7** 435

[28] Wang Y P, Dai J H, Hu Z N and Pu F C 1997 Exact results for a Kondo problem in a one-dimensional *t–J* model *Phys. Rev. Lett.* **79** 1901
(Wang Y P, Dai J H, Hu Z N and Pu F C 1997 *Preprint* cond-mat/9706086)

[29] Zamolodchikov A B and Zamolochikov Al B 1979 Factorized *S*-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models *Ann. Phys.* **120** 253

[30] Zhou H-Q, Ge X-Y, Links J and Gould M D 1998
(Zhou H-Q, Ge X-Y, Links J and Gould M D 1998 *Preprint* cond-mat/9809056)